Tohoku RCG Workshop:

Numbers Game, Minuscule Kac-Moody Representations, Infinite d-Complete Posets

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Part A: Introduction

Here's an outline for my two talks:

After I give an overview in Part A, today and tomorrow I'll be indicating how some special colored posets can connect Lie representation theory to combinatorics.

A1: Graph & Poset Terminology

A1.1 **$** Please allow me begin by establishing some terminology —

In these talks I'll always have one fixed connected graph Γ_n at hand:
it will have n nodes and it will be simple, meaning it won't have loops or multiple edges.

$ I will often refer to its distinct nodes as being distinct "colors":

Often Γ_n will serve as the Dynkin diagram for a Lie algebra.

$ My "directed graphs" may have an infinite number of nodes,
but they also will have no loops or multiple edges.

A1.2 **$** A "poset" is a set with a partial order; the set might be infinite.

The "Hasse diagram" of a poset depicts it with a directed graph.

$ Here's the Hasse diagram for the poset of subsets of the set {1,2,3} when they are partially ordered by containment:

$ Here the subset {1,2,3} <u>covers</u> the subset {1,3} and {2,3} <u>covers</u> {2}.

A1.3 $ In addition to finite posets, we will be considering two kinds of infinite posets.

The foremost "singly infinite" poset is the ordered set of the nonnegative integers:

The foremost "doubly infinite" poset is the ordered set of all integers:

**$** Most of my posets will be "Γ_n-colored";
this will mean that their elements are colored using all of the colors from Γ_n.

For example, given this fixed graph of colors Γ_n, this poset is Γ_n-colored:

A2: Simple Lie Algebras, Weyl Groups, Representations

A2.1 $ The simple Lie algebras over the complex numbers are indexed by the connected Dynkin diagrams:

A2.2 $ The foremost example is sl_{n+1}, which is the vector space of trace-zero $(n+1) \times (n+1)$ matrices equipped with the algebra operation of anti-commutator bracket [.,.].

$ This Lie algebra is a prototype for all simple Lie algebras g, especially the "simply-laced" ones. Their structures are very similar to the stucture of sl_{n+1}.

It will even serve as a prototype for the Kac-Moody Lie algebras, which I will consider later in these talks.

A2.1(!) $ The simply-laced simple Lie algebras are the ones whose Dynkin diagrams do not have multiple edges, namely A_n, D_n, E_6, E_7, and E_8:

These are the simple Lie algebras of Types ADE, which are the ones that I will consider in these talks:

These Dynkin diagrams are connected simple graphs, and so I will denote these diagrams in general with Γ_n.

A2.3 $ Each simple Lie algebra g has several finite structures associated to it,
especially its Weyl group W.

The Weyl group of sl_{n+1} is the symmetric group S_{n+1}; here are the orders of the other Weyl groups of Types ADE:

A2.4 $ Given a vector space V, the Lie algebra gl(V) is the vector space *End*(V) equipped with the anti-commutator bracket [.,.] for linear operators.

A representation φ of a Lie algebra g on V is a Lie algebra homomorphism from g to gl(V).

A2.5(2.4) **$** To describe a representation φ of the simple Lie algebra *g* indexed by the Dynkin diagram Γ_n,

$ We first find the mutual eigenspaces for the actions of its "diagonal" elements.

Each of these eigenspaces has an associated n-tuple of eigenvalues.

These n-tuples are produced by linear functionals on a subspace h of g:

These linear functionals are the <u>weights</u> of j.

So we associate to each simple Lie algebra *g* an n-dimensional vector space *h** that contains the weights of representations of *g*; it is called the <u>weight space</u> of *g*.

$ The Weyl group W for *g* acts on *h**.

A3: Numbers Game

A3.1 **$** The Numbers Game will be my main tool for these talks.

The Numbers Game is played on a fixed connected simple graph Γ_n,
given an initial state λ that is a labeling of its nodes with integers:

For example, here is a labeled simple graph Γ_n that has 7 nodes.

$ For one move,
you can choose any node that has a positive label, say the Red +4 here, and "fire" it —
add that label to the labels of the adjacent nodes and then negate that label.

So here the four labels that are adjacent to the Red +4 label have been increased by 4, and
the Red +4 has been replaced by this Red –4.

(Or I could have fired one of these other two nodes, with the Green +2 or the Blue +1.)

But first, let me note that at this new state I have four positive nodes to choose from for the next firing:

$ If I fire at this Green +2 down here, I will produce the same state as if
$ I had first fired at this Green +2 up here and then secondly fired at the Red +4.

A4: Overview: 5 Realms, Goals, Cases/Flavors

A4.1 **$** In these talks I will be working in 5 different realms:

In Realm-I, I will be concerned with the Numbers Game on a fixed graph Γ_n.

$ In Realm-II,
I will study orbits in *h** of the Weyl group W for the Dynkin diagram Γ_n.

$ In Realm-III, I will study four classses of special colored posets whose Γ_n-colorings obey certain axioms: These are the "minuscule" and "d-complete" Γ_n-colored posets;
each of these kinds of poset can be finite or infinite.

$ In Realm-IV, I will study representations of the Lie algebra *g* for the Dynkin diagram Γ_n that are "built" from Γ_n-colored posets;
these algebras can be simple Lie algebras or Kac-Moody algebras.

$ In Realm-V, I will study representations of the Lie algebra *g* for the Dynkin diagram Γ_n whose weights have certain "nice" properties.

A4.2 **$** My goals in these talks are:

First, within each of the 5 realms, classify the possible structures using Dynkin diagrams.

$ Second, to present the work of Green & Strayer on building Kac-Moody reps;
for this they used the doubly-infinite minuscule posets introduced by Green.

$ Also, I'll introduce a notion of "infinite d-complete" poset.

$ And along the way, I'll review the construction of d-complete posets from the λ-minuscule elements of Kac-Moody Weyl groups.

A5: Cases/Flavors of Structures

A5.1 **\$** Now I'll preview the flavors of the structures that I'll be considering.

We'll always have one fixed connected simple graph Γ_n at hand,
and I'll often view its nodes as being colors.

\$ Early in the talk, if I am talking about Lie algebras,
then Γ_n will be one of the Dynkin diagrams of Types ADE.

Later in the talk, if I am talking about Lie algebras, then Γ_n will be a general simple graph.

Correspondingly, the Lie algebra at hand will be simple (finite dim'l) or Kac-Moody (infinite dim'l).

\$ I will consider only "simply-laced" simple Lie algebras and Kac-Moody algebras.

The directed graphs produced by the Numbers Game can be finite or infinite.

A5.2 **\$** My special Γ_n-colored posets will fall into 4 classes:
minuscule or d-complete, finite or infinite.

I'll explain this table later today.

\$ The minuscule posets will have some nice properties holding in both the "up" and the "down" directions, while the more general d-complete posets will have these nice properties holding only in the "up" direction.

\$ Correspondingly, the representations considered will be for all of g or for just its "Borel" subalgebra $b+$.

A5.3 **\$** Most of the posets P that I'll consider will appear in two versions …
the original poset P itself, but also sometimes its associated lattice L of order ideals.

\$ Each manifestation determines the other manifestation.

\$ Sometimes the larger L-version will appear first;
sometimes the smaller P-version will appear first.

I'll go through the mechanics of the relationship later.

Okay, that completes my introduction …

Part B: Numbers Game Describes Orbits of Weyl Groups

Now we're ready to get to work!

In this part I'll want to indicate how the Numbers Game can be used to describe the orbits of Weyl groups.

$ For Part B, let's fix one Dynkin diagram Γ_n of Type ADE and consider the simple Lie algebra g determined by Γ_n.

B1: Representations of Lie Algebras and Their Weights

B1.1 **$** In the simple Lie algebra g we fix a nice abelian subalgebra h.

It's called the <u>Cartan subalgebra</u>; its dimension n is the <u>rank</u> of g.

$ For example, in the Lie algebra sl_n, we take h to be the subspace of diagonal matrices.

In general, the rank of g is the number of nodes in the Dynkin diagram Γ_n for the algebra.

B1.2 **$** We're now ready to study a representation φ of the Lie algebra g.

To do this, we generalize the notion of "eigenspace" from one linear operator on V to the images of the elements of the subalgebra h on V:

$ Let h^* denote the dual vector space to the CSA h.

A nonzero subspace $U \subseteq V$ is a <u>weight space</u> for φ if there is a linear functional ν on h such that the functional ν produces an eigenvalue for the action of any element from h on any vector u in the subspace U.

Here $\nu \in h^*$ is called the <u>weight</u> for the weight subspace U.

$ So "weight" and "weight space" generalize "eigen-value" and "eigen-space".

Hence the dual space h^* of the Cartan subalgebra h is called the <u>space of weights</u> for g.

B1.3 **\$** For the representation φ, let's collect the weights ν for its weight spaces into a set $\prod(\varphi)$; this is the weight diagram of φ.

We will consider only weight representations;
for these the vector space V can be expressed as the sum of the weight spaces U for φ.

\$ There's a nice basis for the space h^* of weights:

It consists of the fundamental weights $\omega_1, \ldots, \omega_n$,
which are indexed by the nodes of the Dynkin diagram.

A weight λ in h^* is dominant integral if it is a nonnegative sum of the fundamental wts.

\$ A dominant integral weight λ can be described by labeling the Dynkin diagram Γ_n:

Just write the coefficients of λ with respect to the fundamental weights next to the corresponding nodes of the diagram:

B1.4 **\$** The irreducible finite dimensional representations of a simple Lie algebra g have been classified:

Each such representation φ of g has a highest weight λ; this weight is dominant integral.

Up to equivalence, there is one such representation φ for each dominant integral weight λ.

Hence the irreducible finite dimensional representations φ of a simple Lie algebra g are specified by the nonnegative integer labelings λ of its Dynkin diagram Γ_n.

\$ Arbitrary integral linear combinations of the fundamental weights are also important;
these are the integral weights:

B2: Simple Roots, Weyl Group Actions

B2.1 **$** For the weight space *h** there is another important basis $\alpha_1, \ldots, \alpha_n$ of simple roots; these are again indexed by colors i from Γ_n.

$ The space *h** has a nondegenerate bilinear form defined $\langle .,. \rangle$ on it.

For each simple root α_i, there is a simple reflection s_i of the weight space *h**:
For $\mu \in h^*$, set $s_i.\mu := \mu - \langle \mu, \alpha_i \rangle \alpha_i$.

Here the multiple of the simple root α_i to be subtracted from μ is determined by a bilinear form calculation.

$ The Weyl group W of *g* is the subgroup of $GL(h^*)$ generated by these n simple reflections.

For the simple Lie algebras, the Weyl groups are finite.

B2.2 **$** Let's return to considering one representation φ of our simple Lie algebra *g*.

The set $\prod(\varphi)$ of weights for φ is stable under the action of W.

$ Let λ be the highest weight for φ.

Since λ is in $\prod(\varphi)$, we can start to find some of the other weights in $\prod(\varphi)$ by repeatedly acting on it with all applications of the s_i.

$ This was the origin of the Numbers Game (in my 1980 thesis).

B3: Computing Weyl Group Orbits

B3.1 $ I'm now ready to describe Realm-II, which concerns orbits of the Weyl group W:

A W-orbit of a weight λ in *h** can be computed using both the fundamental weight basis and the simple root basis, and by applying two nice facts to the reflection formula:
First, we have $\langle\lambda_j,\alpha_i\rangle = \delta_{ij}$, the Kronecker delta.
Second, the change-of-basis rule from the simple roots to the fundamental weights is nice:
When the Dynkin diagram Γ_n is simply-laced, the rule for the ith simple root is $\alpha_i = 2\omega_i - \sum\omega_j$,
where the sum runs over the nodes j of Γ_n that are adjacent to the node i.

$ Using (1) and (2) in the reflection formula, we see that result of reflecting the fundamental weight ω_j by the simple root α_i is given by the formula $s_i.\omega_j = \omega_j - (2\omega_j - \sum\omega_k) = -\omega_j + \sum\omega_k$,
where the sum runs over all of the nodes k of Γ_n that are adjacent to the node j.

B3.2 $ Let's show how this reflection formula works for the Weyl group E_6:
Here the Node '3' has three adjacent nodes: 2, 4, and 6.
So if we reflect the fundamental weight ω_3 by the simple reflection for α_3 we get the following result:

$ Any integral weight can be displayed with a labeled Dynkin diagrams.
Here are the labeled Dynkin diagrams for the fundamental weight ω_3 and
for the weight we just produced by applying the reflection s_3 to ω_3:

$ For another example, for the Weyl group A_4,
suppose we start out with the dominant weight λ which is just the fundamental weight ω_1.

Here is the result of applying s_1 to ω_1:

The reflection actions of W on *h** are linear.

$ Please allow me to write a '+1' as a '+' sign and a '–1' as a '–' sign.
Then these labeled diagrams show the result of reflecting λ first with respect to α_1,
then with respect to α_2, then α_3, and then finally with respect to α_4:

$ This sequence of results can be viewed as a wave traveling down a straight channel:

B3.3 **$** Let's give a more general example that uses the fact that the reflection actions of W on *h** are linear.

Depict a dominant integral weight λ with a labeled Dynkin diagram, for example D_6:

Suppose we act on λ by the reflection corresponding to a simple root α_i, say α_4.

$ The labels for λ at the nodes that are not adjacent to Node i are unaffected by the action.

The label at i is negated by linearity, and that label is added to the labels of the adjacent labels, again by linearity.

$ This is one move in the Numbers Game on Γ_n!

Iterate these actions in all possible ways. Since W is finite, the Numbers Game generates all of the orbit $W.\lambda$.

B4: Minuscule Representations

B4.1 **$** Continue to consider the simple Lie algebra *g* and the Weyl group W that are specified by our fixed Dynkin diagram Γ_n, and also
consider a representation φ of *g* whose highest weight is the dominant integral weight λ.

The set of weights of φ is now denoted by $\prod(\lambda)$.

It's almost always true that the orbit of the highest weight is not the entire set of weights for a representation: Usually this orbit is a strict subset of the set $\prod(\lambda)$ of all weights.

$ A minuscule representation of *g* is one for which the two sets *are* equal.

The minuscule representations have been classified; their highest weights λ are always fundamental weights.

$ Here are the labeled Dynkin diagrams for the highest weights of the minuscule representations:

B4.2 **$** Let's look at the orbit of one of these minuscule highest weights:

For the Weyl group A_5, let's start with its fundamental weight ω_3:

This directed graph describes the Numbers Game generat'n of the Weyl group orbit of ω_3.

$ Since ω_3 is the highest weight for a minuscule representation,
these weights form the*entire* set of weights $\prod(\lambda)$ for the representation φ.

B4.3 **$** There is a standard partial ordering of the space *h** of weights:

One defines $\mu \leq \nu$ if $\nu - \mu$ is a nonnegative sum of simple roots.

$ When this Weyl group orbit was just computed,
each Numbers Game move subtracted one simple root.

Therefore that directed graph was the Hasse diagram for that orbit set of weights
when they were ordered by the standard partial ordering by roots.

B5: Bruhat Orders, Bruhat Lattices

B5.1 **$** To explain how minuscule and d-complete posets first arose,
I'll need to briefly talk about Bruhat orders.

Let's choose a subset J of the nodes in the diagram Γ_n that specifies the Weyl group W.

Then use the simple reflections s_j for the nodes j in J
to generate a Weyl subgroup W_J of W.

The <u>Bruhat ordering</u> is a partial ordering of the elements of a Weyl group W.

It can also be used to partially order the cosets in W/W_J of the subgroup W_J.

The set of these cosets is denoted W^J.

$ Given the subset J,
there is a dominant integral weight λ_J such that the orbit $W.\lambda_J$ describes W^J.

The Bruhat ordering rule can be translated into this *h** setting.

$ The minuscule representations' highest weights λ are fundamental weights,
which are of the form λ_J for some (n–1)-element subsets J of the nodes in Γ_n.

In these cases the standard "by roots" ordering of the orbit $W_J.\lambda$ by simple roots
is essentially the Bruhat poset W^J. (Actually, it's the order dual of it.)

B5.2 **$** Among the W^J Bruhat orders for the simply-laced groups W: The only distributive lattices are the Bruhat orders that come from the minuscule representations.

$ So this list of labeled Dynkin diagrams also indexes the "Bruhat lattices":

This was the main result of my MSJ plenary talk,
and we've now finished Part B.

Part C: Finite Colored Minuscule Posets

In this part I'll present my first class of Γ_n-colored posets, the finite colored minuscule posets.

C1: Distributive Lattices, Lattices of Ideals

C1.1 **$** First I need to present two more sets of posets definitions:

To start the first collection:

A poset L is a distributive lattice if for any two elements there exists a greatest lower bound and a least upper bound such that the distributive laws are satisfied.

$ The Hasse diagram for a distributive lattice looks like a bunch of d-cubes glued together.

This implies that the covering edges in a distributive lattice can be partitioned into sets of "parallel" edges.

$ An element of a lattice L is defined to be a one-cover if it covers exactly one other element.

I'll denote the subposet of one-covers of L by *OC*(L).

$ Here's a Bruhat distributive lattice W^J for the Weyl group A_4 …
I've circled the subset J of nodes in red, and the highest weight is this λ_J:

It has ten elements.

In this distributive lattice I've circled the one-cover elements:

C1.2 **$** To start the second collection of poset definitions:

In any poset P, an ideal is a subset that is "downwardly closed":
A subset I of P is an ideal if whenever an element y is in I,
then x being less than or equal to y implies that x is in I.

I'll denote the set of all ideals of P by *I*(P);
these can be ordered by containment.

$ For an example, in this red six element poset I've circled the elements of two ideals,
one in blue and one in green.

$ This poset has ten ideals altogether;
when they are ordered by containment the following distributive lattice results:

Notice that this is just the ten element Bruhat lattice $L = W^J$ that we started with!

C2: FTFDL, Minuscule Posets

C2.1 **$** The Fundamental Theorem of Finite Distributive Lattices says that this happens in general — any distributive lattice L can be "distilled" down into its poset *OC*(L) of one-covers, from which L can be recovered by applying the poset-of-ideals construction.

$ Here's a summary of our example:

$ This theorem explains the Confusion that I warned you about in my introduction:

I will often be interested in both the "L" and the "P" manifestations of a poset.

At times the "L" version of a poset will arise first:
Numbers Game graphs, Bruhat distributive lattices, and
at times "P" version of a poset will arise first:
minuscule & d-complete posets.

C2.2 **$** The original Definition of "minuscule" poset was:

The <u>minuscule</u> posets are the posets that arise as posets *OC*(L) of one-covers of the Bruhat distributive lattices W^J for the finite Weyl groups W.

Here they are:

$ These posets have been used to compute the cohomology of minuscule flag manifolds.

C3: Coloring Edges: DL's, Numbers Game, Coloring Minuscule Posets

C3.1 **$** For a moment, consider any fixed connected simple graph Γ_n.

Let's regard its nodes as being colors:

Let P be a Γ_n-colored poset;
recall that this means that the elements of P are colored with the n nodes of Γ_n:

Let's return to the construction of the lattice *I*(P) of ideals of P.

$ The places where a given element is "augmenting" ideals form a set of parallel edges in the Hasse diagram for *I*(P).

$ So the col'ng of P by Γ_n induces a coloring by Γ_n of the sets of parallel edges in *I*(P).

Conversely, each coloring of the sets of parallel edges in a distributive lattice L induces a coloring of the elements of the poset *OC*(L).

C3.2 **$** Now let's consider a Numbers Game played on a Dynkin diagram Γ_n.

Let's regard the distinct nodes of Γ_n as being distinct colors.

Then in the directed graph that describes playing the game, the edges can be colored accord'g to which node is fired.

$ Let's return to the example for A_5, and suppose that this fourth node is pink:

Whenever that node is fired, I have colored that downward edge with pink:

$ In fact, when this is done in any of the Numbers Game directed graphs that generated the Bruhat lattices W^J, all of the edges in a parallel set of edges will receive the same "firing" color.

This Γ_n-coloring of the edges induces a Γ_n-coloring of the elements of the corresponding minuscule poset.

C3.3 **$** I define the resulting colored posets to be the minuscule Γ_n-colored posets; here they are:

C4: Properties of Finite Colored Minuscule Posets, Realm-III

C4.1 **$** These colorings of the minuscule posets have several nice properties.

But first I need to give some more poset definitions:

Let x and y be any two elements of a poset P.

If $x \leq y$, then we say that x and y are comparable.

Two elements are neighbors if one of them covers the other one.

Given $x,y \in P$, the closed interval from x to y is the set of all elements of P that are "weakly" between x and y.

$ A poset is a chain if any two elements in it are comparable.

And for $k \geq 3$ a double-tailed diamond poset (DTD) is a poset of this form:

C4.2 **$** Three of the nice properties that a Γ_n-colored minuscule poset P has are:

(EC) If two elements of P have Equal colors, then they are Comparable.

For example, here all of the Green elements are comparable.

$ (NA) The colors of two Neighboring elements in P must be Adjacent in the Dynkin diagram Γ_n for P.

For example, the neighbors of the Pink elements in this poset are only of the colors green, brown, and light blue.

$ (DTD) If x and y are consecutive occurrences of a given color, then the interval [x,y] is a Double-Tailed-Diamond.

For example, the intervals between the first and second blue elements and between the second and third blue elements are both Double-Tailed-Diamonds.

C4.3 $ As I continue to work in Realm-III, I will be concerned with four special kinds of Γ_n-colored posets: posets that have the "full strength" minuscule properties or that merely have the weaker "d-complete" properties.

These posets may be finite or infinite. So far today I have presented the finite minuscule Γ_n-colored posets.

I introduced the minuscule posets and d-complete posets in 1984 and 1999, and Green introduced the infinite minuscule posets in 2007.

$ A student of mine who is graduating this year, Michael Strayer, has produced some nice uniform definitions that encompass all four classes of these Γ_n-colored posets.

These definitions impose coloring requirements such as the three properties EC, NA, and DTD.

That's it for Part C; we're now ready to

Part D: K-M Weyl Groups, λ-Minuscule w's, d-Complete P's

work more generally in the context of Kac-Moody Weyl groups.

D1: Kac-Moody Weyl groups

D1.1 $ To produce the finite min'le posets, it was good enough to work with the finite Weyl groups of Types ADE.

But to produce the other three classes of Γ_n-colored posets that I am interested in,
it is necessary to work with the messier Kac-Moody algebras and their infinite Weyl groups.

This is true even for the *finite* d-complete posets.

So for the rest of these talks:

From now on let's fix *any* connected simple graph Γ_n that has n nodes.

Every such graph Γ_n specifies a simply-laced K-M Weyl group W.

The group W is infinite when Γ_n is not of Type ADE.

I'll associate two vector spaces to the group W:
the space h^* of weights and an n-dimensional quotient vector space $h^{*\prime}$ of h^*.

As before, I'll associate vectors $\omega_1,\ldots,\omega_n$ and $\alpha_1,\ldots,\alpha_n$ to the nodes of Γ_n.

The ω_i form a basis of $h^{*\prime}$.

$ The $\alpha_1,\ldots,\alpha_n$ no longer give a basis of all of the vector space h^*, and now the bilinear form $\langle .,. \rangle$ on h^* might be degenerate!

I'll denote their images in $h^{*\prime}$ by $\alpha_1', \ldots, \alpha_n'$

As before, we will be concerned with the integral linear combinations of the ω_i.

Again I will call these <u>integral weights</u>; they are depicted with the integral labelings of Γ_n.

D1.2 **$** We still have the ingredients needed to use the Numbers Game to model the action of the Weyl group W, provided that we now work in h'^*.

Let μ be an integral weight in h'^*. Set $s_i.\mu := \mu - \langle \alpha_i,\mu \rangle \alpha_i'$,

$ Still have the nice two facts:

(1) $\langle \alpha_i,\lambda_j \rangle = \delta_{ij}$. (So this evaluation $\langle \alpha_i,\mu \rangle$ is the label of the weight μ at node i in Γ_n.)

(2) Change-of-basis: $\alpha_i' = 2\omega_i - \sum\omega_j$,
where the sum runs over the nodes j of Γ_n that are adjacent to the node i.

$ But in the Kac-Moody case the Numbers Game modeling of the action of W is not faithful; we have to now work harder.

D2: λ-minuscule elements

D2.1 **$** Let $\lambda \in h^{*\prime}$ be an integral weight.

Dale Peterson defined an element w of a Kac-Moody Weyl group W to be <u>λ-minuscule</u> if it can be expressed as w $= s_{i_k}\ldots s_{i_1}$ for some $k \geq 0$ and some $i_1, \ldots, i_k \in \Gamma_n$ such that
$\langle s_{i_{j-1}}\ldots s_{i_1}.\lambda,\alpha_{i_j} \rangle = +1$ for $1 \leq j \leq k$.

$ This means that the simple root α_{i_j} itself is being subtracted at the jth stage;
not some multiple of it.

It can be shown that $s_{i_k}\ldots s_{i_1}$ is a reduced decomposition, and
that any other reduced decomposition for w satisfies this definition.

D2.2 **$** We can restrict the Numbers Game to find the λ-minuscule elements of W:

Starting with the labeling λ of Γ_n, we can now fire a node only if it is labeled with a +1.

$ We can repeat this to increase the "length" of the w that we are creating, for as long as there is at least one +1 label present on Γ_n.

Each such firing sequence of +1's will create a λ-minuscule element w of W.

D2.3 **$** To illustrate this definition, let's look at part of playing the Numbers Game for the Weyl group E_6 with this initial labeling λ:

$ Now if the nodes in the Dynkin diagram are numbered in this way,
I'll label the firing edges with the numbers of the diagram nodes.

Since all of the firings here are +1's, at each stage one simple root was being subtracted, and
so we read down any firing sequence from the initial state λ and write down the simple root reflections from right to left, then we will have created a reduced decomposition for the λ-minuscule element w.

$ These red arrows indicate one particular firing sequence, which leads
to this particular reduced decompositon for one λ-minuscule Weyl group element w:

D2.4 **$** For some Kac-Moody Weyl groups, this picture is not as easy:

Suppose we choose this weight for the affine $\widetilde{E}_6$ to be the initial state for a Numbers Game:

If we apply the following sequence of λ-minuscule firings (this is just part of the Numbers Game's directed graph), then we end up back at our initial state λ!

But we can still use this graph to create λ-minuscule elements of W.

$ This first sequence describes a reduced decomposition of one λ-minuscule Weyl group element w_1, and if we repeat this sequence of firings, then we obtain a reduced decomposition for a second λ-minuscule element w_2.

But ideally: It would be nice for the states in the Numbers Game directed graph to correspond bijectively to the λ-minuscule elements w.

D3: Augmented Numbers Game

D3.1 **$** To obtain a faithful depiction of a Weyl group orbit,
I augment the labelings of Γ_n with "tallies".

In the Augmented Numbers Game on Γ_n the states are now ordered pairs:
The first component is a labeling of Γ_n and the second component is the tally n-tuple.

$ The tally n-tuple for the initial state is defined to be **0**.

For the later states,
the tally n-tuple records how many times each color has been fired so far.

$ Now the cycling phenomenon can't occur.

D3.2 **$** Let's return to our $\widetilde{E}_6$ example, but now let's add in the tally n-tuples:

These two states are now distinct.

Within combinatorics it can be seen that this directed graph is acylic.

As far as depicting elements of the Weyl group is concerned:

Using the linear independence of the α_i in h^*, it can be shown that distinct λ-minuscule elements w will create distinct augmented states.

So we can use the Augmented #'s Game to faithfully construct as much of the Weyl group orbit at λ as we want.

D4: Wave Theorem, d-Complete Posets

D4.1 **$** Let's return to considering the Bruhat ordering on the Weyl group W for Γ_n.

The identity element e of W is the unique minimal element in this ordering.

Each element w of the group W specifies a "principal" ideal [e,w] of this Bruhat poset.

$ Now fix an integral weight λ and one λ-minuscule element w.

Let's define the Poset of Tallies for w to consist of all of the tallies that can be produced "on the way" from λ to $w.\lambda$, ordered by componentwise comparison.

D4.2 **$** Here's the main result of a 1999 paper of mine; it generalizes the creation of the minuscule posets from the Bruhat lattices for the finite Weyl groups:

Let λ be a dominant integral weight. Let w be a λ-minuscule element of W.

(1) The poset of tallies assigned to w is isomorphic to the ideal [e,w] in the Bruhat order.

(2) This poset is a distributive lattice.

$ (3) The poset of one-covers in [e,w] is a Γ_n-colored poset that has a set of "d-complete" coloring properties.

(4) Every Γ_n-colored poset with the d-complete properties arises in this fashion.

$ A d-complete Γ_n-colored poset was defined to be a Γ_n-colored poset that has these coloring properties.

D4.3 **$** The original "d-complete" properties included
the nice properties EC, AC, and DTD noted earlier for the minuscule posets.

$ So I've now indicated how both the finite minuscule and the finite d-complete posets arose from Bruhat orders on Weyl groups.

We're now ready to begin to discuss

Part E: Infinite Minuscule & d-Complete Posets, Classifications

Infinite minuscule & d-complete posets and their classifications.

E1: Extended Numbers Game

E1.1 **$** Now I want to indicate how infinite minuscule and d-complete posets will hopefully arise from the Numbers Game; this is work-in-progress.

As before, let's fix some connected simple graph Γ_n and a labeling λ of it.

I'll now refer to λ as our "reference" state (and not as our "initial" state);
its tally n-tuple is still (0, 0, …, 0).

There is another way in which I'll extend the original Numbers Game.

$ In addition to augmenting the states to produce an acyclic directed graph,
I'll now also allow "anti-firing" moves:

At any node with a negative label in any state, one can "anti-fire" that node by reversing the usual firing action:

Add that label to the labels of the adjacent nodes and then reverse the sign of that label.

Diminish the component for that node in tally n-tuple by adding that negative label to it.

The resulting state may or may not be present in the directed graph constructed so far.

If it isn't present, then adjoin this new state to the directed graph.

Draw the arrow in the directed graph in the opposite direction.

$ Here I illustrate this on the initial (now reference) state for our $\tilde{E}_6$ example:

I'll anti-fire the '–1' to produce this new state; this diminishes that component of the tally:

E1.2 **$** Here's a portion of our $\tilde{E}_6$ example near the same reference state λ:

The directed graph for this Extended Augmented Numbers Game is doubly infinite …
it goes on forever in both the down and the up directions.

As before, the (undisplayed) tally n-tuples make the states with the same labels distinct.

E1.3 **$** Is this infinite directed graph the Hasse diagram for an infinite distributive lattice?

If it is, now that we're working with an infinite lattice L,
can it still somehow be produced by the *I*(.) functor from its subposet *OC*(L)?

$ There are other examples of game graphs that look similar to this … all of their labels come from {+1,0,–1}.

$ If they are distributive lattices L, can we characterize their posets *OC*(L) of one-covers as being Γ_n-colored posets that satisfy certain properties?

Since these Numbers Games' directed graphs are so similar to those for the minuscule lattices, doing this could lead to a notion of "infinite minuscule" Γ_n-colored poset.

However, Green was led to a notion of "infinite minuscule" Γ_n-colored poset from another direction, which I'll soon describe.

E2: Strayer's Uniform Definition of Finite and Infinite Minuscule Posets

E2.1 $ In a 2001 paper, John Stembridge extended my theorem for λ-minuscule Bruhat lattices and d-complete posets to Kac-Moody Weyl groups that are not simply-laced.

When doing so, he developed a some nicer axioms to define finite d-complete posets.

In a 2007 paper, Richard Green used some of Stembridge's axioms when he formulated a notion of infinite minuscule poset.

But to define finite minuscule posets, he had to give a separate definition.

$ Let's return to the 2×2 table that summarizes the history for the Γ_n-colored posets we are considering.

$ Recently my student Michael Strayer improved upon Green's definition of infinite minuscule posets so that he could define finite minuscule posets at the same time.

$ So now the definition of minuscule poset can be made uniformly for finite and infinite posets at the same time.

E2.2 $ Before presenting Strayer's definition,
we need to introduce a common assumption in combinatorics:

A poset is <u>locally finite</u> if every interval [x,y] in it is finite.

Here's Strayer's definition:

A locally finite Γ_n-colored poset P is <u>minuscule</u>
if its coloring is such that the following six Γ_n-coloring properties hold:

E2.3 $ Earlier, I introduced these first two properties for finite minuscule posets:

(EC) any two elements in P with the same color are comparable, and

(NA) any two neighboring elements in P have distinct colors that are adjacent in Γ_n.

$ Stembridge and Green introduced the next two properties; the second of these is closely related to the DTD interval property I introduced before:

(AC) any two elements in P whose colors are adjacent in Γ_n are comparable, and

(I2A) strictly between any two consecutive occurences of a given color in P, there exist exactly two elements whose colors are adjacent to that color in Γ_n.

$ Strayer developed the last two properties:

(Mx1GA) the number of elements above a maximal element of a given color whose colors are adjacent to that color in Γ_n cannot exceed 1.

(Mn1LA) the number of elements below a minimal element of a given color whose colors are adjacent to that color in Γ_n cannot exceed 1.

E2.4 **$** Here are two examples of infinite minuscule Γ_n-colored posets (but without the colors):

If we form the lattice of ideals for the second one, then we we'll produce the directed graph for the $\widetilde{E}_6$ example of the Extended Numbers Game above.

E3: Strayer's Uniform Definition of Finite and Infinite d-Complete Posets

E3.1 **$** In 2015, when revisiting d-complete posets with my student Scoppetta,
it seemed that there should also be a notion of "d-complete" for locally finite posets.

But I was not able to come up with a "good" definition of this concept.

**$** Strayer's definition of "Γ_n-colored minuscule" is easily modified to become a good definition of "Γ_n-colored d-complete" posets:

One simply drops the requirement of Axiom MN1LA:

A locally finite Γ_n-colored poset P is <u>d-complete</u> if its coloring is such that
Axioms EC, NA, AC, I2A, Mx1GA are satisfied.

So this definition also does not refer to the cardinality of P;
it is the *first* definition of d-complete poset that "works" for infinite posets.

What are some examples of infinite d-complete Γ_n-colored posets?

$ First we need another poset definition:

A subset F of a poset P is a <u>filter</u> if it is "upwardly closed", namely:

For every $y \in F$, if $x \geq y$ then $x \in F$.

It is easy to see that any filter (possibly the entire poset) of a minuscule poset is a d-complete poset.

E3.2 **$** So for examples of infinite d-complete posets,
we can take any filters of the two infinite minuscule posets that I just displayed

$ Here are those filters as posets in their own right:

E4: Classifications of Finite Minuscule & d-Complete Posets

E4.1 **$** Let's check back in with the my "big view" of these talks:

Now that I've presented the definitions for all four kinds of my posets,
I have introduced all of the structures considered in Realms I, II, and III.

It's not hard to see that Realms I and II are equivalent.

Also, I've indicated how the *finite* minuscule and *finite* d-complete posets arise from orbits of the Weyl group.

$ I now want to talk about the four problems of listing all of the possible structures in each of the five realms.

E4.2 **$** It's known that the only possible finite minuscule Γ_n-colored posets are the five kinds that I displayed earlier:

$ So at this point it's natural to seek to list all of the other three kinds of posets:

E4.3 **$** Building upon the 2007-2014(?) work of Green and his student McGregor-Dorsey, and upon my own work of 1984 and 1999, Strayer and I can now list all possible Γ_n-colored posets in each of the four classes: minuscule and d-complete, finite and infinite.

E4.4 In 1999 I classified the finite d-complete posets.

$ These Γ_n-colored posets were produced by playing the Augmented Numbers Game starting from the following initial labelings λ on the following graphs:

Although these posets are finite, most of these graphs correspond to infinite Kac-Moody Weyl groups, not to finite Weyl groups.

$ The "slant irreducible components" of the finite d-complete Γ_n-colored posets fall into 15 classes.

Here from one initial labeling λ the Augmented Numbers Game can often be played in "different directions".

But in for every sequence of choices, there is a maximal possible finite game graph.

The following displayed posets are the corresponding maximal possibilities —

E4.5&6&7 **$** Here are the one-cover posets for the maximal λ-minuscule Bruhat lattices that are generated from these labeled Y-shaped graphs Γ_n:

E5: Classifications of Infinite Minuscule & d-Complete Posets

E5.1 **$** McG-D finished Green's classification of the infinite minuscule posets in 2013.

$ All of their possibilities came from some of the affine Kac-Moody Weyl groups;
here are the Dynkin diagrams for those groups:

McGregor-Dorsey developed a system to index the possibilities.

I won't present his indexing system here.

E5.2 **$** Instead, I'll propose listing the possible infinite minuscule Γ_n-colored posets as follows: I conjecture that the following labeled simply-laced affine Dynkin diagrams list possible reference labelings from which one can use the Extended Augmented Numbers Game to generate all of the distributive lattices for the infinite minuscule Γ_n-colored posets.

E5.3 **$** Finally, Strayer and I have shown that if a Γ_n-colored poset is d-complete,
then it must be a filter of one of the infinite minuscule posets.

$ So all four of our kinds of special Γ_n-colored posets have now been classified,
which completes Part E.

Part F: Building Representations from Colored Posets

In this part I want to describe the original motivations for considering infinite minuscule and d-complete posets.

F1: Splits

F1.1 $ For the time being, I am done with Realms I, II, and III … so we'll have a fresh start in Realm IV for Part F!

Building representations from colored posets is the topic that provided the motivation to Green and Strayer for considering infinite minuscule and d-complete posets.

$ First, I need some more poset definitions —
I want to develop a new way of viewing the ideals of P.

Observe that if $I \subseteq P$ is an ideal, then the set $P - I$ is a filter of P.
And if $F \subseteq P$ is a filter, then the set $P - F$ is an ideal of P.

So the filters and the ideals of P occur in complementary pairs (F,I).

$ Considering such a pair (F,I) is equivalent to splitting the Hasse diagram of P into two parts.

So let's define a split of P to be a filter-order pair (F,I) such that F and I partition P.

F1.2 $ Let's denote the set of all splits of P by $\mathit{FI}(P)$.

When it is ordered by the inclusion of the ideals in the splits,
it is just the distributive lattice $\mathit{I}(P)$ that was considered before.

$ Let's return to the lattice of ideals of that six element poset.

Let's look at one split (F,I).

$ If z is a minimal element of F, then $(F-\{z\}, I\cup\{z\})$ is a split.

So "transfering" a minimal element of F to the ideal I forms another split.

F2: Raising & Lowering Operators

F2.0 **$** Fix a graph Γ_n of colors and a Γ_n-colored poset P.

Let V(P) be the free complex vector space on the set of splits of P.

The basis vector corresponding to (F,I) is denoted $\langle F,I \rangle$.

$ Choose one color i from Γ_n.

Define a "raising" linear operator X_b on V(P) by taking the result of it acting on a split $\langle F,I \rangle$ to be the sum of all splits $\langle F',I' \rangle$ such that there exists some element minimal element z of F of color i such that $F' = F - \{z\}$ and $I' = I \cup \{z\}$.

Dually-analogously define a "lowering" linear operator Y_i on V(P); it sums over all of the ways of transferring maximal elements in I of color i to the complementary filter F.

$ Continuing with our example, let's suppose that this element 'e' of the poset has color '4'.

Then applying X_4 to this split will describe the result of transfering that element of color '4' from this filter F to this filter I:

$ Finally, on V(P) set $H_i := X_i Y_i - Y_i X_i$.

F3: Internal Structures of Simple Lie Algebras

F3.1 **$** Every simple Lie algebra g has a structure that is similar to that of sl_{n+1}.

$ The Dynkin diagram A_n of sl_{n+1} has n nodes:

To each node $i \in A_n$ we associate two matrices $x_i := E_{i,i+1}$ and $y_i := E_{i+1,i}$ in sl_{n+1}.

Using the commutator bracket [.,.], this Lie algebra is generated from these 2n elements.

And these 2n elements satisfy certain relations in bracket.

Let V be a vector space.

$ If we can define 2n linear operators X_i and Y_i for $i \in A_n$ which satisfy the analogous relations in the commutator bracket for V, then sending $x_i \mapsto X_i$ and $y_i \mapsto Y_i$ will specify a representation of sl_{n+1} on V.

F3.2 $ Let's continue our example.

Let's color the six elements of our poset with the four colors from this Dynkin diagram, which is the Dynkin diagram for sl_{4+1}.

Let's construct the four raising operators X_1, X_2, X_3, X_4 and the lowering operators Y_1, Y_2, Y_3, Y_4.

$ It can be seen that these eight linear operators on the vector space generated by the ten splits here satisfy all of the relations satisfied by the generators x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 for the Lie algebra sl_{4+1}.

Therefore these eight operators generate a representation of the algebra sl_{4+1} on this ten-dimensional vector space.

$ The brackets $[x_i,y_i]$ produce important elements; these are equal to $E_{i,i} - E_{i+1,i+1}$, and these elements are given the names h_i for $i \in A_n$.

F4: Simply-Laced Reduced Kac-Moody g' for Γ_n

F4.1 $ Now let's return to our fixed general connected simple graph Γ_n with n nodes.

For each color $i \in \Gamma_n$, create two symbols x_i and y_i.

Use these 2n symbols to generate a free Lie algebra over $\mathbb{C}$.

For each color i in Γ_n, set $h_i := [x_i,y_i]$.

$ The simply-laced reduced Kac-Moody algebra g' specified by Γ_n is the Lie algebra produced by imposing the following relations on this free Lie algebra:

F4.2 These new Lie algebras g' include the simply-laced simple Lie algebras, but they are infinite dimensional whenever Γ_n is not A_n, D_n, or E_6, E_7, E_8.

$ As for sl_{n+1}, if on some vector space V we can define 2n linear operators X_i and Y_i for $i \in \Gamma_n$ which satisfy the analogous relations in the commutator bracket for End(V), then sending $x_i \mapsto X_i$ and $y_i \mapsto Y_i$ will specify a representation of g' on V.

F5: Theorems of Green and of Strayer

F5.1 **$** Now let's also fix a Γ_n-colored poset P.

Construct its lattice *FI*(P) of splits and the free vector space *V*(P) on *FI*(P).

Construct the raising and lowering operators X_i and Y_i on *FI*(P).

$ We want these 2n operators to generate a representation φ of the reduced Kac-Moody algebra *g*´ for Γ_n.

When this happens, we say that φ has been <u>built</u> from P.

$ Green used his doubly infinite minuscule posets to build doubly infinite representations of affine Kac-Moody algebras, and he remarked that these representations "looked like" the minuscule representations of simple Lie algebras.

F5.2 **$** Strayer defined a representation of *g*´ built from a Γ_n-colored poset P to be <u>P-minuscule</u> if all of the eigenvalues for all of the H_i are ±1 or 0.

The minuscule representations of the simple Lie algebras also have all of their eigenvalues equal to ±1 or 0, and the labels in the most interesting Numbers Games are all ±1 or 0.

Recall that Strayer improved Green's definition of "minuscule" poset.

$ Here's the main result of Strayer's thesis:

Let Γ_n be a connected simple graph.

A P-minuscule representation of the derived Kac-Moody algebra *g*´ for Γ_n can be built from a locally finite Γ_n-colored poset P if and only if P is a minuscule Γ_n-colored poset.

$ Strayer's result holds uniformly for finite and infinite minuscule posets.

This completes my presentation of the representation theory motivation for considering infinite posets.

Part G: Classifying Infinite Minuscule Structures

In this last part of my talks I'd like to describe some current and future work with my student Michael Strayer; we want to classify all of the kinds of structures that I've been talking about yesterday and today.

G1: Minuscule Abstract Representations of Kac-Moody Algebras?

G1.1 **$** I would like to now begin to present Realm-V:

Can we develop an "abstract" notion of a minuscule representation for Kac-Moody algebras?

By this I mean one that does not refer á priori to a poset.

The minuscule representations of the simple Lie algebras are widely known.

They are usually defined in terms of their highest weights.

Sometimes it is said: "For the Kac-Moody algebras that aren't the simple Lie algebras (the infinite ones) there are no minuscule representations."

$ Green's representations "look like" the minuscule representations of simple Lie algebras in many ways.

But they don't have highest weights.

Strayer's notion of "P-minuscule" refered to a poset P that was already on hand.

G1.2 **$** But what if there is not already a poset P at hand?

Problem. Formulate a definition of "minuscule" for "abstract"representations of Kac-Moody algebras.

This definition should refer only to pre-existing representation theory concepts.

$ We propose the following definition:

Def'n. A weight representation φ of a Kac-Moody algebra g is minuscule if:

(1) all of its weight spaces are 1-dimensional,

(2) its weight diagram $\prod(\varphi)$ under the standard root ordering is a connected poset, and

(3) all of the eigenvalues for the images of the algebra generators h_i acting on the weight vectors for φ are ±1 or 0.

When the algebra g is a simple Lie algebra, this definition re-produces the usual minuscule representations.

$ We believe we can prove that Green's representations are the only infinite dimensional Kac-Moody representations that satisfy this definition.

G2: Overview of All Classifications, Three Classification Problems

G2.1 **$** Can these minuscule abstract representations of Kac-Moody algebras be classified?

How about the the infinite minuscule structures in our other four realms …
can they be classified?

If so, would the statements and proofs of these five classifications be related to each other?

$ In all five realms the *finite* minuscule structures have been listed.

In Realm-III all of the infinite Γ_n-colored minuscule posets have been listed;
I presented the list earlier.

In Realm-IV, Strayer can apply the Realm-III classification to list all of the possible P-minuscule representations of a Kac-Moody algebra g.

$ Strayer and I are currently classifying the possible infinite minuscule Γ_n-colored structures
in the other three realms: I, II, and V.

G2.2 **$** To state our conjectured classifications, I need a few more definitions:

A set of integers is tiny if it is a subset of {+1,0,–1}.

A labeling λ of Γ_n is tiny if its set of labels is tiny.

An integral weight $\lambda \in h^{*\prime}$ for the Weyl group for Γ_n is tiny if its coefficients with respect to the fundamental weights are tiny.

$ I'll now state the classification problems that Strayer and I are currently working on:

Problem-I: List every graph Γ_n with a tiny labeling such that the labelings arising in the Extended Augmented Numbers Game from that tiny reference labeling are tiny.

Problem-II: List every graph Γ_n with a tiny labeling such that the labelings in the Kac-Moody Weyl group orbit based at that tiny labeling have only tiny labels.

Problem-V: Classify the abstract minuscule representations of the Kac-Moody algebra g for Γ_n.

G3: Conjecture-Theorems and Their Equivalence

G3.1 $ We believe we will soon finish the proofs of the following three statements:

The only "tiny" structures that can arise in Problems I, II, and V are those that can generated from the labeled affine Dynkin diagrams Γ_n that I presented earlier:

G3.2 $ In Realms I and II, any tiny labeling/integral weight that arises can be used as the reference labeling or orbit "base point".

In Problem V, we choose a "zero level" for a certain element of the Cartan subalgebra *h*.

$ It is easy to see that Problems I and II are equivalent using a Kac-Moody version of the relationship described yesterday for finite Weyl groups.

Showing the equivalence of Problems II and V appears to be straightforward,
using some standard facts from Kac-Moody representation theory.

$ So we will only need to solve Problem-I, for the Numbers Game.

Okay, so we're now done witht the entire outline!