MSJ Regional Tohoku Meeting:
How are combinatorial games related to Lie theory!?

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February 16, 2019

A0 **$** I know of two combinatorial games that are used in Lie representation theory.

Both are one player combinatorial games without any random aspects;
playing such a game is visualized by forming a path in a directed graph.

$ For an example, consider the child's tile game —
it could start out at this state, and then
$ at each stage you can slide any adjacent tile into the currently empty space:

So at the beginning there are four possible moves:

And at each later state there would be two, three, or four possible moves.

B1 Before I get to my main topic, I'd like to briefly digress —

$ Schützenberger developed "jeu de taquin" to prove the Littlewood-Richardson rule,
which is used to compute:

the tensor product of two representations of GL_n, which is also
the product of two Schur functions, which is closely related to
the cup product in the cohomology of the Grassmannian manifolds.

$ Skew Schur functions can be used for these computations:
each one is described with some reverse skew semistandard Young tableaux.

For the Littlewood-Richardson rule, we need to "rectify" this skew tableau.

To visualize this process,
let's rotate this skew tableau and view its numbers as being Red Weights;
then let's put some Green Bubbles underneath the weights in the empty boxes.

Now let's let the Green bubbles rise.

B2 **$** But we need to choose an order for releasing the bubbles.

First, let's let the left-top bubble be the first one to rise — denote it with a Green '1':
Since the Red '6' is heavier than the Red '4', it will sink into the location of Bubble #1.
Then the Red '3' is heavier than the Red '2', and so it sinks next.//

$ When Bubble #1 reaches the top, it disappears … Poof!
Then this higher of the two remaining bubbles has to be the next one to rise — call it #2.

$** Repeating for it and **$ for the last bubble, we end up with this non-skew tableau:

$ On the other hand, let's let the top bubble to the right be the first one to rise:
The red '5' sinks into its place, and we repeat such steps Poof! Poof! Poof! to again produce a rectified tableau.

But look … Both bubble orders produced the same straight tableau!

This remarkable uniqueness is a key result.

C1 **$** Okay, now I'll start on my main topic …

A Numbers Game is played on a fixed simple graph Γ_n that has n nodes,
given an initial state that is a labeling of its nodes with integers.

For example, here is a labeled simple graph that has 7 nodes.

$ For one move,
you can choose any node that has a positive label, say the Red +4 here, and "fire" it —
add that label to the labels of the adjacent nodes and then negate that label.

So here the four labels that are adjacent to the Red +4 label have been increased by 4, and
the Red +4 has been replaced by this Red –4.

(Or I could have fired one of these other two nodes, with the Green +2 or the Blue +1.)

But first, let me note that at this new state I have four positive nodes to choose from for the next firing:

$ If I fire at this Green +2 down here, I will produce the same state as if
$ I had first fired at this Green +2 up here and then secondly fired at the Red +4.

(This process is <u>not</u> commutative when two adjacent nodes are fired.)

C2 **$** Today I will show how the Numbers Game can be used to answer the following question: Which Bruhat orders are distributive lattices?

$ These partially ordered sets arise in Lie theory and algebraic geometry; distributive lattices are a very nice kind of partially ordered set.

D1 **$** The complex simple Lie algebras are indexed by the connected Dynkin diagrams:

Today I will be concerned with only the "simply-laced" cases, those of Types A, D, E; their Dynkin diagrams Γ_n are connected simple graphs:

D2 **$** Each simple Lie algebra has a Weyl group associated to it; these are finite groups with the following orders:

E0 **$** A Poset is a Finite set with a Partial ordering.

$ For example, to produce a poset with 8 elements,

$ One could order the subsets of a 3-element set by containment:

Here we say that the subset {1,2,3} covers the subset {1,3} and the subset {2,3} covers the subset {2}.

F1 **$** The elements of a Weyl group are Partially ordered by the Bruhat ordering to produce the Bruhat orders (or Bruhat posets).

For example, here are the Bruhat orders on the Symmetric group S_3 and on the Hyperoctahedral group which is the group of symmetries of the square:

The Identity element is always the Minimum element.

F2 **$** The Bruhat ordering is also defined on some sets of cosets of Weyl groups:

Given a Dynkin diagram Γ_n, let Γ_n also denote the set of its nodes.

Each subset J of Γ_n specifies a sub-Weyl group W_J of W.

The Bruhat order is extended to the set of cosets W/W_J, which is denoted W^J.

$ A diagram Γ_n has 2^n subsets of its nodes.

Today I'm considering the Weyl groups for the Diagrams A_n, D_n, E_6, ,E_7, and E_8.

For each n there are 2^n Bruhat orders of types A and D, and
there are $2^6 + 2^7 + 2^8$ Bruhat orders of type E.

The problem that I'm considering in this talk is:

Which of the Bruhat orders on these sets of cosets are Distributive lattices?

F3 **$** Let me give an example of the Bruhat ordering on a set of cosets:

In the Dynkin diagram A_5, suppose we choose this subset J of nodes.

There will be 90 cosets in W^J.

$ They are depicted with the shuffles of two 1's, two 2's, and two 3's;

Here's a start to drawing the diagram for this poset W^J:

F4 **$** The Bruhat order on W^J was invented to describe the containments of the Schubert varieties in the flag manifold G/P_J formed from the Lie group G of type Γ_n.

For type A_{n-1}, the most notable of these are the Grassmannians and the full flag manifold G/B.

$ The Numbers Game was developed to describe the Bruhat orders.

G1 **$** Next, I need to tell you what a distributive lattice is:

A poset L is a distributive lattice if for any two elements there exists a greatest lower bound and a least upper bound such that the distributive laws are satisfied.

An element of a lattice L is a one-cover if it covers exactly one other element.

The subposet of one-covers of a poset L is denoted *OC*(L).

$ And here's another batch of poset definitions:

A subset of a poset is an ideal if it is "downwardly closed":

Let *I*(P) denote the set of ideals of a poset P.

This is itself a partially ordered set, by containment.

G2 **$** Here's a test to see if a poset is a distributive lattice:

A poset L is a distributive lattice if and only if it is isomorphic to the poset of the order ideals of its subposet of one-covers.

Corresponding to this Red Subset J of the Dynkin diagram A_4,
here is the Bruhat order W^J, which I'll call 'L', of the shuffles of two 1's and three 2's:

$ First I'll circle its One-Covers in Green; here is the Subposet *OC*(L) consisting of these One-Covers:

Please let me rename these six elements with the Red letters a, b, c, d, e, f.

Finally, I'll form the Red Poset of Ideals in this One-Covers poset:

(For example, this ideal circled in Blue becomes this element {b a c d} in the Red Poset.)

Since we have re-produced the original poset L, that poset is a Distributive lattice!

H1 **$** Here is the answer to today's question:

The only Bruhat orders W^J of Types A, D, or E that are distributive lattices are those for the following (n–1)-element subsets J:

For the diagram A_n we can choose J to be any subset of that size.
For the diagram D_n we can choose J to be one of only these three subsets:

$ And the distributive lattices are the posets $I(P)$ of ideals for the posets P shown here:

H2 **$** There are no Bruhat orders for the Weyl group E_8 that are distributive lattices.

For the Weyl groups E_6 and E_7 there are respectively only two possible subsets J and one possible subset J:

$ And those distributive lattices are the posets $I(P)$ of ideals for the posets P shown here:

The posets indicated on these two slides are by definition called the minuscule posets;
they were used by Thomas & Yong to extend the Littlewood-Richardson rule for computing cohomology from the Grassmannian to some other flag manifolds.

I1 **$** Here's the Numbers Game Tool that is used to prove this theorem:

Let Γ_n be a simply-laced Dynkin diagram. Let J be a subset of Γ_n.
The Bruhat order W^J can be obtained from the directed graph of states in the Numbers Game on the graph Γ_n, starting with a certain labeling λ_J.

$ For nearly all of the choices of subsets J, the Numbers Game can be used to produce a local substructure that can't exist in a distributive lattice.

I2 **$** Let me give an example of this:
For A_5, choosing this J leads to the maximum shuffle being 3,3,2,2,1,1.

Corresponding to this shuffle is this Initial Labeling λ_J,
which I'll write as 0, 1, 0, 1, 0.

$ Suppose we fire this first '1':

Doing this will lead to having two 1's that are adjacent.

Firing the nodes at these two 1's will produce two further states,
and from them we can produce two more states.

$ The ordering rule for the Bruhat order implies that these blue lines also describe covering relationships:

But this criss-cross substructure can't be in a lattice!

All of the Bruhat orders that aren't distributive lattices can be ruled out like this.

I3 **$** On the other hand, here's an example of a Bruhat poset that is a distributive lattice:

For E_6 this subset J has this Initial Labeling λ_J, which I'll write like this:

After firing the +1, the first new state is this, which I'll write like this:

$ As we play the Numbers Game in all possible ways, the following directed graph arises:

$ I've circled the one-covers here in red.

This red subposet can be used to see that this Bruhat W^J <u>will</u> pass the distributive lattice test!

J1 **$** Here's why the Numbers Game describes a poset W^J of cosets:

Recall from linear algebra that one linear operator is described
with its eigenspaces and eigenvalues.

$ To describe a representation of the Simple Lie group for the Dynkin diagram Γ_n,
look at the images of the elements in an n-parameter "Torus" abelian subgroup T of it:

Simultaneously Diagonalize these images.

Their Mutual eigenspaces are the <u>weight spaces</u> for the representation.

$ The logarithms of the eigenvalues in the n-tuples of eigenvalues for the weight spaces
form the <u>weight vectors</u> for the representation.

These live in an n-dimensional Euclidean space **E**.

These vectors also help to describe the corresponding representation of the simple Lie algebra for the diagram Γ_n.

J2 **$** There is a basis for **E** that consists of n special vectors;
these are the <u>simple roots</u> for Γ_n.

The Weyl group for Γ_n acts on **E** by reflecting it with respect to these vectors.

$ Let's specify a Bruhat order W^J by choosing a Subset J of the nodes in the diagram Γ_n.

Create an initial weight λ_J for W^J
by forming the sum of the "fundamental" weights for the nodes not in J.

The orbit $W.\lambda_J$ of this initial weight describes the set W^J of cosets.

J3 **$** Next, create an initial labeling for the Numbers Game
by placing 0's by the nodes in J and 1's by the other nodes of Γ_n.

This initial labeling describes the initial weight λ_J.

$ Now firing a node in the Numbers Game describes reflecting with respect to a simple root!

$ So the states in the Numbers Game are the weights in the orbit $W.\lambda_J$ of λ_J.

In the distributive lattice cases,
the Numbers Game directed graphs describe the Bruhat orderings.

K1 **$** The Numbers Game be used to produce some other interesting posets,
which are called "d-complete" posets.

Recall that the Minuscule Posets were the One-Cover subposets of the Bruhat lattices for the Weyl groups for the Simple Lie algebras.

$ Now consider the Weyl Groups for the Kac-Moody Lie algebras.

Here none of the sets of cosets W^J are finite,
but every coset wW_J in W^J specifies a finite Bruhat Order.

Some of the simplest Kac-Moody Weyl group cosets come from the "Lambda-Minuscule" elements.

$ Using the Numbers Game, I showed that the Bruhat orders for the Lambda-Minuscule cosets are Distributive lattices, and that their subposets of One-Covers satisfy some "d-Complete" Axioms.

K2 **$** The posets that satisfy these d-complete axioms, which are called "d-complete" posets, can be shown to possess some nice combinatorial properties.

Let's fix a positive integer k.

Let $p_k(n)$ be the number of partitions of n into at most k parts.

Euler found this Generating Function Identity for $p_k(n)$:

$ Stanley extended this concept of counting partitions to the context of any Poset Q.

He said a poset Q is <u>hook length</u> if its generating function has a Product Identity of the same beautiful form as in Euler's identity:

Here a positive integer h(y) is being assigned to each element y of the poset, and
the product runs over the elements of the poset Q.

$ The following two results were announced in Kyoto in 1998:

Every d-Complete Poset is a Hook Length Poset.

Dale Peterson used algebraic geometry to help me to prove this result.

Also, I was able to show that
Every d-Complete Poset has unique Jeu de Taquin rectifications.

K3 $ Ishikawa and Tagawa defined some posets that generalized d-complete posets;
they call their posets "Leaf" posets.

They proved that their Leaf posets also had the Hook Length property.

If a Poset P is d-Complete,
then its Lattice I(P) of Ideals can be generated with the Numbers Game.

$ Can a version of the Numbers Game be developed that generates the Lattice I(P) of Ideals for any Leaf Poset P?

$ Some of the other people who have worked with d-Complete Posets include:

Kawanaka, Nakada, Okamura, Okada, Naruse, Stembridge, Ram, Kleshchev, R.M. Green, Chaput, Perrin, Kim, Yoo, D. Peterson, and Hagiwara.

L1 $ Today I showed how to use the Numbers Game to create Bruhat lattices and their associated minuscule posets from finite Weyl groups, and also the more general d-complete posets from Kac-Moody Weyl groups.

On Monday and Tuesday I'll describe a more general version of the Numbers Game
that allows you to also play "backwards" with "anti-firings":

L2 $ Here's a slide from Tuesday's talk that shows an anti-firing:

L3 $ Doing firings and anti-firings in all ways will generate a "doubly infinite" directed graphs:

L4 $ And the associated poset (on the right) will also be doubly infinite:

L5 $ So I will be talking about four classes of posets …
finite and infinite minuscule and d-complete posets:

L6 $ Associated to the Bruhat lattices are the important "minuscule" representations of simple Lie algebras.

$ It has not been known how to define "minuscule" representations of Kac-Moody Lie algebras.

$ My goal in my second talk will be to introduce a good definition of "minuscule" for representations of Kac-Moody algebras.